Optimum Design of Stressed Skin Structures

GEOFFREY G. POPE*
Royal Aircraft Establishment, Farnborough, Hampshire, England

Applications of the move limit method are described in the least weight design of elastic stressed-skin structures, using finite elements, when the design variables consist of member cross sections and skin thicknesses. Numerical examples presented involve up to two load conditions with active constraints on stresses and displacements and on the design variables themselves which are subjected to upper and lower bounds. Convergence is compared where appropriate with that of a fully-stressed design iteration. Computational efficiency achieved confirms the usefulness of the sequence of linear programs approach.

I. Introduction

THE power of the digital computer in the analysis of complex structures is now universally recognized and the associated computational techniques have reached an advanced stage of development. The computer can also be used in the automation of more general portions of the design process when these consist of repetitive sequences of computations interspersed with decisions that are taken on a factual rather than a subjective basis. The useful range of such applications is increasing as more efficient hardware, software and numerical techniques become available; benefits achievable may include an acceleration of the design process, a more efficient deployment of design effort and an improved final design resulting from a more thorough examination of possible alternatives.

One class of such applications, which is the subject of this paper, is the choice of reinforcing member cross sections and skin thicknesses in stressed-skin elastic structures of specified basic geometry which are represented by finite element idealizations. Optimum designs are sought to carry several distinct loadings in turn when constraints are imposed on behavioral variables such as stresses and displacements and on the permissible range of the design variables themselves. In aeronautical applications optimality is usually synonymous with least weight once the over-all configuration and the structural material have been selected (fail safe features can be looked upon as behavioral constraints), so attention is restricted here to least weight design, although the approach employed can be used with any merit function with continuous first derivatives with respect to the design variables. For simplicity, weights of joints and connections are omitted but these may readily be incorporated.

Techniques of two distinct types have received vigorous development in recent years for use in this context. Firstly there are techniques based on the general methods of mathematical programing, which are reviewed in Pope and Schmit,[1] and secondly there are those based on optimality criteria which have been evolved for use in specific classes of application; an approach of the latter type may be firmly established when constraints on displacement only are involved[2] but applications of this[3] and similar approaches[4,5] to airframe structures usually include an element of semi-intuitive reasoning when constraints on stresses are active.

The mathematical programing techniques converge in general to designs which are certainly local optima and which often prove to be global optima; repeat computations starting from radically different initial designs may be used, if necessary, to verify to a reasonable level of confidence that a global optimum has indeed been obtained. Such verification is, however, seldom undertaken in practice on account of the high cost of the associated computations, and it is better to view these methods as powerful means of improving designs derived on a less systematic basis rather than as techniques for finding truly optimal designs.

The main advantage of the optimality criterion approach is that convergence often proves more rapid than that of existing mathematical programing algorithms, but in many circumstances the criteria adopted are not well enough established to guarantee that an optimum design has been obtained; repeat computations from differing starting points are of limited value if the optimality criterion may itself be deficient.

The purpose of this paper is to describe applications to stressed-skin structures of one particular mathematical programing technique, the move limit method,[6] which is based on the reduction of nonlinear programing problems to sequences of problems in linear programing. In structural design applications where constraints on displacement or stiffness are active as well as constraints on stress, convergence is shown to be sufficiently rapid to make this more rigorous approach a useful alternative to techniques based on optimality criteria. The move limit method remains applicable, moreover, when more general classes of constraint and merit function are employed which are beyond the scope of optimality criterion methods. Earlier applications to simpler types of structures are described by Reinschmidt, Cornell, and Brotchie,[7] by Moses,[8] by Pope[9] and by Romstad and Wang.[10] The present paper implements and develops an approach outlined previously[11] by the author for use when two-dimensional stress fields are involved.

Sections II and III, respectively, outline the basis of the move limit method and set up the structural design problem specified above in a form suitable for solution with its aid. Section IV describes numerical examples which have been computed, and conclusions drawn from these are presented in Sec. V.

II. The Move Limit Method

II.1 Basic Theory

This section summarizes the basic theory of the move limit method; a broader background is included in Pope and Schmit.[1] If a vector **D** is used to specify the values of the design variables needed to define the structure and if M represents the merit function, which is here chosen to be weight, then structural design problems of the class considered in this paper may be expressed in the following form:

Presented as Paper 73-342 at the AIAA/ASME/SAE 14th Structures, Structural Dynamics, and Materials Conference, Williamsburg, Va., March 20–22, 1973; submitted March 29, 1973; revision received July 12, 1973.

Index categories: Structural Design, Optimal; Aircraft Structural Design.

* Previously Superintendent, Research Division, Structures Department; presently Deputy Head (Tunnels), Aerodynamics Department.

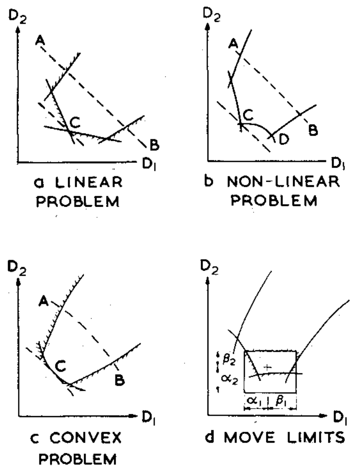

Fig. 1 Typical situations in two variable problems.

Find **D** such that

$$h_k(\mathbf{D}) \leq 0; \quad k = 1, 2, \ldots, K \quad \text{and} \quad M(\mathbf{D}) \to \min \tag{1}$$

The functions $h_k(\mathbf{D})$ express constrained quantities, including those of a behavioral nature, in terms of the design variables. They are seldom all known explicitly since it is usually difficult and unrewarding to obtain expressions for behavioral quantities such as stresses and displacements purely in terms of the design variables, i.e., it is difficult to eliminate variables originating in the structural analysis. It is convenient nevertheless to consider the problem conceptually in the aforementioned form and to focus attention initially onto situations involving two variables only.

Consider first a problem in which the functions h_k and M are all linear and in which the design is specified by two variables D_1 and D_2. The design problem then reduces to a problem in linear programing and may be illustrated as shown in Fig. 1a where the full lines represent the equations $h_k(\mathbf{D}) = 0$ which define the edges of the feasible region, and the broken line AB indicates the locus of designs for which the merit function has an arbitrary constant value. The design which minimises M while satisfying all the constraints is located at C, the point of contact of a line parallel to AB which just touches the feasible region and which passes between it and the origin. If AB happens to be parallel to a constraint on the relevant boundary of the feasible region, then the point C is not unique and there is a whole range of optimal designs. Usually, however, the point C lies at a vertex of the feasible region, irrespective of the number of design variables, provided that the constraints and the merit function are linear.

When either a constraint or the merit function is nonlinear, the point C will not necessarily lie at a vertex of the feasible region (see Fig. 1b) and local optima may exist (e.g., point D). The latter difficulty does not arise in problems where the constraints and merit function have the forms illustrated in Fig. 1c; such problems, which are known as convex problems, cannot, however, be identified easily in the structural design context on account of the large numbers of variables and constraints that are usually involved.

If it were known definitely that the optimum design C in a nonlinear problem occurred at a vertex of the feasible region, the following simple sequence of linear programs method[12] could be adopted: 1) Linearise the constraints and the merit function in the neighbourhood of an arbitrary design $\mathbf{D}_0$ and solve the resulting linear programing problem. 2) Replacing $\mathbf{D}_0$ with the optimum solution $\mathbf{D}_q$ obtained in the preceding linear problem, repeat the process until no significant change occurs in the solutions obtained from successive linear problems.

If the optimum design does not lie at a vertex of the feasible region, successive designs obtained by this procedure would oscillate indefinitely between the vicinities of vertices lying on the critical constraint or constraints. Convergence may, however, be achieved by adding to the individual linear problems move limits of the form

$$\mathbf{D}_q - \boldsymbol{\alpha} \leq \mathbf{D} \leq \mathbf{D}_q + \boldsymbol{\beta} \tag{2}$$

as illustrated in Fig. 1d where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are vectors of suitably chosen positive constants. The values of these constants are reduced progressively on the basis of some criterion such as that described in Sec. II.2, until the optimum design is located to the desired degree of accuracy.

II.2 Details of Procedure Adopted in the Present Application to Least Weight Structural Design

Relatively large values of the constants contained in the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ must be chosen initially so that the imposed move limits do not impede convergence to the vicinity of the optimum solution. It might be advantageous to reduce the individual constants by differing amounts when oscillation occurs, depending on the role played by the associated variable. In the examples described in this paper, however, all such constants were chosen, for simplicity, to be the same fractions of the corresponding variables in the design used as a starting point in the relevant linearized computation, i.e.

$$\boldsymbol{\alpha} = \boldsymbol{\beta} = x\mathbf{D}_q \tag{3}$$

The factor x was chosen initially to be 0.5 and was halved each time a reduction in the move limits proved necessary. It should be noted that a different definition of the move limits not related to the variable itself would be needed on any variable which is likely to tend to zero, if acceptable convergence is to be achieved.

In describing the precise procedure adopted in the reduction of the move limits it is convenient first to consider structural applications in which the constraints consist solely of lower bounds on the design variables and upper bounds on the absolute values of displacements and stresses. In such applications negligible computational effort is involved in factoring the design $\mathbf{D}_q$ obtained in a typical linearized computation to achieve a design $\gamma_q \mathbf{D}_q$ which is just feasible. The next linearized computation is started from the design $\gamma_q \mathbf{D}_q$ and gives rise to a new design $\mathbf{D}_{q+1}$ which is in turn factored to give a just feasible design $\gamma_{q+1} \mathbf{D}_{q+1}$. Comparing the weights of these designs, a reduction in the move limits is needed if $\gamma_{q+1} M_{q+1} > \gamma_q M_q$. It would, however, be computationally inefficient to reject the direction of search yielded by the linearization about design $\gamma_q \mathbf{D}_q$ purely because the distance of travel has proved excessive. Instead the weights are examined of a series of just feasible designs $\gamma_{q+r+1} \mathbf{D}_{q+r+1}$, which are obtained by analyzing and factoring designs given by

$$\mathbf{D}_{q+r+1} = 0.5(\gamma_q \mathbf{D}_q + \mathbf{D}_{q+r}), \quad r = 1, 2, 3, \quad \text{etc.} \tag{4}$$

As soon as a design $\mathbf{D}_{q+n+1}$ is obtained such that $\gamma_{q+n+1} M_{q+n+1} \leq \gamma_q M_q$ this step halving process is terminated and the sequence of linear programs procedure is re-entered using $\gamma_{q+n+1} \mathbf{D}_{q+n+1}$ as the initial design, with the move limits reduced to $\frac{1}{2}^n$ times their previous amplitude. Usually only a single application of Eq. (4) is needed to achieve a lighter feasible design.

In applications where upper bounds are imposed on the design variables, the above move limit reduction and redesign procedures continue to be used; the just feasible designs must, however, be replaced by "semi-feasible designs" which may violate this particular class of constraint but are in other respects just feasible. When the move limits become tight, a situation can arise where a linearization about a "semifeasible" design gives rise to a linear program which has no feasible solution. Whenever such a linear program is formulated a new linearization is initiated about a revised design in which a typical design variable D_{ir} is given by the following expressions:

$$\begin{aligned} D_{ir} &= (1 + 0.5x) D_{i(r-1)}, \quad D_{ir} \leq D_i^+ \\ D_{ir} &= D_i^+, \quad (1 + 0.5x) D_{i(r-1)} \geq D_i^+ \end{aligned} \tag{5}$$

$D_{i(r-1)}$ indicates the corresponding value in the design about which the preceding linearization was attempted, D_i^+ indicates the upper bound on the variable and x is defined in Eq. (3). Once this situation has arisen it proves preferable to perform subsequent linearizations directly about the design obtained as the solution to the preceding linear programing problem. Move limits continue to be tightened, however, on the basis of the comparative weights of successive "semifeasible" designs and the initial design from which the sequence of linear programs is recommenced after a tightening of the move limits continues to be based on Eq. (4).

The procedures described previously are carried out automatically within a computer program developed by the author which also incorporates the associated structural analysis described in Sec. III. Criteria could be incorporated in this computer program to terminate the computations when the weight reduction being achieved in just feasible and "semifeasible" designs is negligible and when upper bounds imposed on the design variables are satisfied to the required accuracy. Such automated criteria were not employed in this investigation, however, since it proved more convenient to perform a specified small number of cycles in each computer run.

III. Equations and Inequalities Defining the Structure and its Deformation

III.1 Basic Formulation

The structural configuration is represented by a finite element idealization, the deformation of which is defined completely by the displacement components at a finite number of grid points. The displacement components at the grid points due to a typical load system j are expressed as a column matrix $\mathbf{r}_j$ which excludes components that are prescribed zero to eliminate rigid body movements. A column matrix $\mathbf{R}_j$ is used to represent the corresponding forces which are related to the displacements by an equation of the form

$$\mathbf{K}\mathbf{r}_j = \mathbf{R}_j \tag{6}$$

where $\mathbf{K}$ represents the relevant stiffness matrix. For simplicity it is assumed here that the applied loads are known explicitly whenever the associated displacements are not constrained to be zero.

Displacement components that are subjected to positive or negative limits on their permissible values are denoted, respectively, by $\mathbf{r}_j^\alpha$ and $\mathbf{r}_j^\beta$ where

$$\left.\begin{aligned}\mathbf{r}_j^\alpha &= \mathbf{C}_j^\alpha \mathbf{r}_j\\ \mathbf{r}_j^\beta &= \mathbf{C}_j^\beta \mathbf{r}_j\end{aligned}\right\} \tag{7}$$

and where $\mathbf{C}_j^\alpha$ and $\mathbf{C}_j^\beta$ are simple transformation matrices. The components $\mathbf{r}_j^\alpha$ and $\mathbf{r}_j^\beta$ may include not only grid point displacements but also generalized displacements which are linear functions of these. For example, torsional constraints in wing design could be expressed in terms of the differences between relevant grid point displacements. Denoting the extreme permissible values of these constrained components by $\mathbf{r}_j^{\alpha+}$ and $\mathbf{r}_j^{\beta-}$, the complete constraints on the displacements may be expressed in the form

$$\begin{aligned}\mathbf{r}_j^{\alpha+} &\geqq \mathbf{C}_j^\alpha \mathbf{r}_j\\ \mathbf{r}_j^{\beta-} &\leqq \mathbf{C}_j^\beta \mathbf{r}_j\end{aligned} \tag{8}$$

The finite elements used to represent the structure are defined in such a way that yielding must necessarily start at their ends or corners. The deformation of the idealized structure remains purely elastic, therefore, provided that the stresses at a finite number of reference points are such that the corresponding values of an appropriate equivalent stress σ lie within the permissible range. It is assumed here that yielding is governed by the von Mises criterion which in a plane stress situation is given by[13]

$$\sigma = (\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2)^{1/2} \tag{9}$$

where σ_x, σ_y, and τ_{xy} are the local values of the stress components. The equivalent stresses at the reference points are expressed as a column vector $\boldsymbol{\sigma}_j$ when a load system j is applied, so the condition that the structure does not yield under this loading is given by

$$\boldsymbol{\sigma}_j \leqq \boldsymbol{\sigma}^+ \tag{10}$$

where $\boldsymbol{\sigma}^+$ is the corresponding vector of critical values. Note that the von Mises criterion prescribes equal yield stresses in tension and compression.

The design parameters specifying the cross section of one-dimensional members and the thickness of plane stress elements constitute the vector $\mathbf{D}$ of the design variables which are, in general, subject to the constraints

$$\mathbf{D}^- \leqq \mathbf{D} \leqq \mathbf{D}^+ \tag{11}$$

where $\mathbf{D}^-$ and $\mathbf{D}^+$ represent vectors of the least and largest permissible values.

III.2 The Linear Subproblem

The basic equations and inequalities are now linearized using the suffix 0 to denote symbols relating to the precisely-analyzed design which is used as the starting point for a typical linearization of the equations.

Let the symbol $\mathbf{Q}_j$ denote a rectangular matrix, the ith column of which is formed by taking the partial derivative of the stiffness matrix $\mathbf{K}$ with respect to the ith design parameter, and by postmultiplying the resulting matrix by $\mathbf{r}_j$. If the initial values of the design parameters are modified by a small increment $\boldsymbol{\delta}\mathbf{D}$, it follows from Eq. (6) that the corresponding change $\boldsymbol{\delta}\mathbf{r}_j$ in the displacements due to a load j is given by

$$\mathbf{K}_0 \boldsymbol{\delta}\mathbf{r}_j + \mathbf{Q}_{j0} \boldsymbol{\delta}\mathbf{D} = 0$$

or

$$\mathbf{K}_0 \mathbf{r}_j + \mathbf{Q}_{j0} \boldsymbol{\delta}\mathbf{D} = \mathbf{K}_0 \mathbf{r}_0 \tag{12}$$

The maximum and minimum permissible values of the design parameters during a typical linearized computation are denoted respectively by column vectors $\mathbf{d}^+$ and $\mathbf{d}^-$, typical elements of which are defined in the following expressions:

$$\begin{aligned} d_i^+ &= (1+x)D_{i0}, \quad D_i^+ > (1+x)D_{i0}\\ &= D_i^+, \quad (1+x)D_{i0} \geqq D_i^+ \end{aligned} \tag{13}$$

$$\begin{aligned} d_i^- &= (1-x)D_{i0}, \quad (1-x)D_{i0} > D_i^-\\ &= D_i^-, \quad D_i^- > (1-x)D_{i0} \end{aligned} \tag{14}$$

where d_i^+, d_i^-, D_i^+, D_i^- and D_i are corresponding elements of $\mathbf{d}_i^+$, $\mathbf{d}_i^-$, $\mathbf{D}_i^+$, $\mathbf{D}_i^-$ and $\mathbf{D}_i$, and where the move limit factor x [Eq. (3)] is chosen to be the same for all elements.

The equivalent stress σ at a typical reference point is related to the initial value σ_0 by a linearized relation of the form

$$\sigma = \sigma_0 + \Sigma \left(\frac{\partial \sigma}{\partial \sigma_{rs}}\right)_0 \delta\sigma_{rs} \tag{15}$$

where the summation includes all the stress components σ_{rs} at the reference point. Substituting Eq. (9) in Eq. (15), the following expression is obtained for the plane stress elements:

$$\sigma = \sigma_0 + \frac{1}{2\sigma_0}\{(2\sigma_{x0} - \sigma_{y0})\delta\sigma_x + (2\sigma_{y0} - \sigma_{x0})\delta\sigma_y + 6\tau_{xy0}\delta\tau_{xy}\} \tag{16}$$

Since elastic deformations only are considered, the stresses may readily be expressed in terms of the strains which may, in turn, be expressed in terms of the displacements at the grid points. Therefore, when the structure is modified slightly, the values of the equivalent stresses $\boldsymbol{\sigma}_j$ may be related to the relevant displacements by an expression of the form

$$\boldsymbol{\sigma}_j = \boldsymbol{\sigma}_{j0} + \mathbf{A}_{j0} \boldsymbol{\delta}\mathbf{r}_j \tag{17}$$

where the coefficients of the matrix $\mathbf{A}_{j0}$ may be deduced from the stress-strain relations and the displacement patterns assumed in the finite elements.

The merit function is related, in general, to the design variables by the expression

$$M = M_0 + \mathbf{M}_0^* \boldsymbol{\delta}\mathbf{D} \tag{18}$$

where $\mathbf{M}^*$ is a row vector of the derivatives of M with respect to the design variables. In all the examples considered in this paper M is, in fact, a linear function of these variables.

III.3 Solution of the Linear Subproblem

It is convenient to express the vector of the design variables

$\mathbf{D}$ in terms of a vector $\mathbf{D}^p$ whose elements are so defined that their minimum permissible values are zero, i.e.,

$$\mathbf{D}^p = \delta\mathbf{D} + \mathbf{D}_0 - \mathbf{d}^- \tag{19}$$

Substituting this expression in Eq. (12), we obtain

$$\mathbf{r}_j = \mathbf{L}_{j0}\,\mathbf{D}^p + \mathbf{N}_{j0} + \mathbf{r}_{j0} \tag{20}$$

where

$$\mathbf{L}_{j0} = -\mathbf{K}_0^{-1}\mathbf{Q}_{j0}, \quad \mathbf{N}_{j0} = -\mathbf{L}_{j0}(\mathbf{D}_0 - \mathbf{d}^-) \tag{21}$$

Substitution of Eq. (20) in Eq. (17) gives

$$\boldsymbol{\sigma}_j = \boldsymbol{\sigma}_{j0} + \mathbf{A}_{j0}\,\mathbf{L}_{j0}\,\mathbf{D}^p + \mathbf{A}_{j0}\,\mathbf{N}_{j0} \tag{22}$$

The linearized problem therefore may be expressed in terms of the design variables only, as follows

$$\text{minimise} \quad M = M_0 + \mathbf{M}_0^*(\mathbf{D}^p - \mathbf{D}_0 + \mathbf{d}^-) \tag{23}$$

subject to the constraints

$$\begin{aligned} \mathbf{D}^p &\leqq \mathbf{d}^+ - \mathbf{d}^- \\ \mathbf{A}_{j0}\,\mathbf{L}_{j0}\,\mathbf{D}^p &\leqq \boldsymbol{\sigma}^+ - \boldsymbol{\sigma}_{j0} - \mathbf{A}_{j0}\,\mathbf{N}_{j0} \\ \mathbf{C}_j^\alpha \mathbf{L}_{j0}\,\mathbf{D}^p &\leqq \mathbf{r}_j^{\alpha +} - \mathbf{C}_j^\alpha(\mathbf{N}_{j0} + r_{j0}) \\ -\mathbf{C}_j^\beta \mathbf{L}_{j0}\,\mathbf{D}^p &\leqq -\mathbf{r}_j^{\beta -} + \mathbf{C}_j^\beta(\mathbf{N}_{j0} + r_{j0}) \end{aligned} \tag{24}$$

where j takes all values from 1 to J, the total number of loadings considered.

For practical merit functions such as weight the elements of the row matrix $\mathbf{M}_0^*$ are positive. Thus a basic feasible solution to the dual of this problem may be obtained directly by choosing the slack variables to be the nonzero variables.[10] The Simplex method is used to find the optimum solution to the dual problem and consequently to the primal problem as well. If no feasible solution exists to the primal problem the merit function in the dual problem can take an indefinitely large value. In the applications described in this paper such a situation can only arise when upper bounds are imposed on the design variables; the technique then adopted within the computer program to find a more appropriate design about which to linearize the analysis is described in Sec. II.2.

III.4 The Finite Element Idealization and Related Computer Program Details

The finite element idealization employed in this investigation consists of triangular membrane elements in which the strains and thickness vary linearly and axially-loaded flanges along which axial strain and cross-sectional area vary linearly. Stiffness matrices for these elements are given in analytical form by Argyris[14–16] together with the associated relationships between element stresses and nodal displacements from which the matrices $\mathbf{A}_{j0}$ may be assembled.

When appropriate the thickness of a membrane or the cross-sectional area of a reinforcing member at a grid point in the finite element idealization is treated as a single design variable by prescribing the thicknesses or cross-sectional areas of the relevant adjacent elements to be equal at the grid point. The corresponding element equivalent stresses and their derivatives are then averaged and a single stress constraint formulated. Note that this facility must be used with caution when coarse grid idealizations are employed and when significant stress discontinuities arise between elements.

Since the design problem is formulated purely in terms of the cross-sectional areas of axially-loaded members and the thicknesses of membrane elements, both the weight M of the structure and the stiffness matrix $\mathbf{K}$ are linear functions of the design variables. Consequently the derivatives of M and $\mathbf{K}$ with respect to these variables are constants and need be computed once only. In the interest of computational efficiency the matrix $\mathbf{K}$ is stored in the form of diagonal bands, taking full advantage of symmetry and using the minimum bandwidth necessary with the chosen node numbering to include all nonzero terms. Similarly only the smallest blocks of rows needed to include all nonzero terms are stored of the derivative matrices $\partial\mathbf{K}/\partial D_i$. The analysis equations are solved using an in-core Choleski decomposition algorithm; the inverse of $\mathbf{K}$ is not computed explicitly. The derivative matrices $\partial\mathbf{K}/\partial D_i$ are stored on exchangeable disk files and are retrieved whenever the matrices $\mathbf{K}$ and $\mathbf{Q}$ require recomputation; transformation matrices necessary to assemble the matrices $\mathbf{A}_j$ are also computed once only and stored on disk files.

IV. Numerical Examples

The practical value of any technique for optimum design depends on its efficiency in comparison with alternative methods, so the applications of the move limit method described below have been studied in some detail to gain insight into its convergence properties. The structural geometries and idealizations considered have been kept simple in the interest of economy, but the associated design problems nonetheless include many features relevant to practical problems in the sizing of aircraft structural elements. It should be noted, however, that since the idealizations employed here are relatively crude, numerical values obtained for the design variables must be interpreted with caution.

When appropriate, comparisons are made between the convergence of the move limit method and that of a fully-stressed design iteration using the same structural analysis computer program segments, the same initial design, the same Fortran compiler and the same computer (ICL 1904A or 1907). In the latter redesign procedure the structural elements are re-sized after each analysis in such a way that each would be fully-stressed under at least one of the applied loadings if the associated changes in stiffness did not influence the loads applied to the individual elements; if any element thickness or cross section obtained in this manner proves to be less than the minimum permissible value, then it is replaced by the latter value.

Convergence of the move limit method is examined by plotting the weights of feasible and semi-feasible designs (see Sec. II.2) obtained from successive linearizations of the basic design problem as a function of central processor computing time. The resulting series of points are joined by straight lines in the relevant figures for clarity of presentation. Note that whenever a new design is heavier than its predecessor, the new design is abandoned and the redesign process continued as indicated in Sec. II.2, using tighter move limits. Consequently, there is a horizontal segment in the convergence curves for the sequence of linear programs method every time the move limits are halved. Note that the "loose end" rising from the beginning of such segments in Figs. 3 and 7 indicates the weight increase obtained when the move limits were retained at their previous value. Each computation was initiated with the move limit parameter $x = 0.5$; the relevant figures indicate the initial value of this parameter over the weight range plotted. Since the time taken to generate a new design in a fully-stressed design iteration is substantially less than the time taken over a single linearized computation in the move limit method, the individual designs obtained by the former approach are merged into a smooth curve and the redesign cycle time is indicated in a caption.

It is difficult in numerical studies to decide at what stage to terminate the search for an optimum design. In the examples

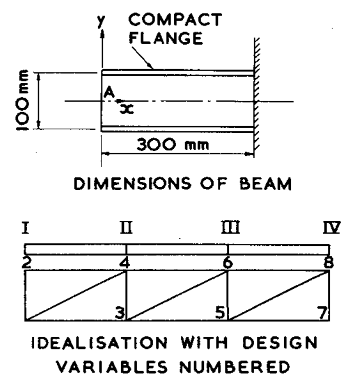

Fig. 2 Beam example—dimensions and basic idealization.

Table 1 Optimum distribution of material in beam

	Case 1		Case 2			Case 1		Case 2	
Variable number	Initial mm	Optimum mm	Initial mm	Optimum mm	Variable number	Initial mm^2	Optimum mm^2	Initial mm^2	Optimum mm^2
1	11.92	6.65	16.03	10.41	I	0	0	0	0
2	(all plane	1.22	(all plane	4.94	II	398	115	534	168
3	stress	2.49	stress	5.77	III	398	291	534	724
4	variables	8.07	variables	14.52	IV	398	480	534	1160
5	equal)	2.83	equal)	5.22	volume of material in half beam $(m^3) \times 10^{-3}$	0.2782	0.1498	0.3741	0.2959
6		8.86		14.21					
7		1.25		4.94					
8		9.92		14.28					

described here where the purpose is to examine the characteristics of the optimization procedure it is desirable to continue further than one normally would in a practical design context, and accordingly computations were continued until the weight reduction over two successive linearizations was less than 0.05%; corresponding changes in some of the design variables were, however, frequently an order of magnitude greater. This tends to confirm that in practical applications a designer may often be able to choose between a range of designs of virtually the same weight to suit requirements not incorporated in an idealized formulation of the optimum design problem.

IV.1 Simple Beam

Consider first the design of a simple cantilever beam consisting of a thin web and two compact flanges as shown in Fig. 2. The beam is to be made from an aluminum alloy for which Young's modulus is 69 GN m^{-2} and the von Mises equivalent stress at yielding is 200 MN m^{-2}; it is to be capable of carrying in turn without yielding two separate loadings consisting firstly of a normal in-plane concentrated load of 40 kN applied on the web centerline at the free end (point A), and secondly of a uniformly distributed normal in-plane load of 180 kN m^{-1}, half of which is applied to each flange. Least weight designs are required (Case 1) when no displacement constraints are imposed and (Case 2) when the displacements at point A are not to exceed half their values under the same load conditions in Case 1. No limitation is imposed on the permissible range of the design variables.

The idealization used to represent the beam is shown in Fig. 2; note that as the beam and the loading are respectively symmetrical and antisymmetrical about the centerline, only half of the beam need be considered. The uniformly distributed loading is represented by point forces which are calculated on a virtual work basis and which are applied at the nodes of the grid and at the midpoints of the relevant element sides.[16] Excluding displacement components that are prescribed zero at the built-in end and along the axis of symmetry, the finite element analysis involves 30 variables; the design is specified in terms of 12 design variables including the cross-sectional area at the free end of the flange which was assumed to be zero since, from consideration of equilibrium, the end load there is necessarily zero. This assumption was subsequently validated in a brief ancillary computation.

The search for the optimum design was started in each case from a design in which all members are uniform, with the exception of the element at the free end of the flange. Initial and optimum designs are given in Table 1; both displacement constraints are active in the optimum design for Case 2. Computing times are shown in Fig. 3 together with those of a fully-stressed design iteration for Case 1. Note that the latter process converges to a design which is 1.6% heavier than the optimum design obtained by the move limit method. A difference between these two designs is hardly surprising in this instance since the flange which is in a state of pure tension cannot be fully-stressed at the same time as the immediately adjacent web which has the same equivalent yield stress and elastic properties and which is in a combined state of direct and shear stress. Consequently an exactly fully-stressed design is an impossibility in this and in many similar situations in stressed-skin structures where members are sized in terms of a finite number of design variables. Note, however, that differences between fully-stressed and optimum designs in the absence of displacement constraints are not entirely attributable to discretization of the structure, the three bar truss under two separate loadings[1] being a striking example of a problem where the two designs can differ by a significant amount (7%) which is not attributable to this source.

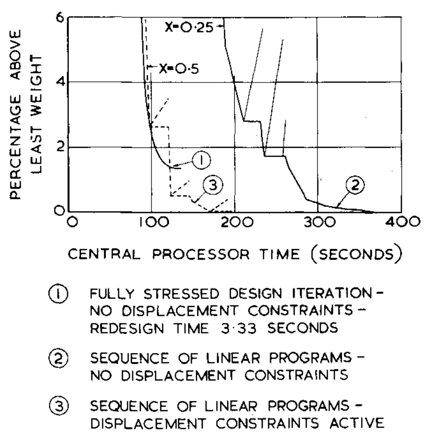

Fig. 3 Beam example—computing times.

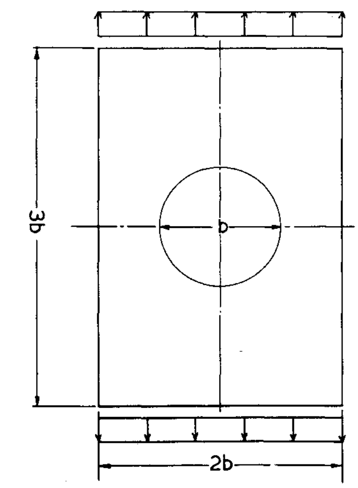

Fig. 4 Plate with hole, under uniform end load.

Table 2 Optimum distribution of material in plate

Variable number	t/t_0 (plane stress variables) or B/bt_0 (flange variables) (t and B denote thickness and cross-sectional area respectively)							
	Case 1		Case 2		Case 3		Case 4	
	Initial	Optimum	Initial	Optimum	Initial	Optimum	Initial	Optimum
1	4.739	1	4.739	1	3.348	1	5.500	1
2	(all	1	(all	1	(all plane	1	(all plane	1
3	variables	1	variables	4.739	stress	1	stress	1
4	equal)	1	equal)	4.739	variables	1	variables	1.458
5		18.795		4.739	equal)	5.500	equal)	13.535
6		1		1		1		1
7		1		1.415		1		1
8		1		4.739		1		1
9		1.930		4.739		1.715		3.471
10		19.295		3.691		7.075		15.355
11		5.945		2.256		4.989		18.085
12		1		1.233		1		1
13		1		3.454		1		1
14		1.369		4.739		1.255		4.801
15		3.479		1.293		3.295		7.175
16		1.278		1.010		1.309		8.060
17		1.255		1.952		1.261		1.721
18		1.239		2.018		1.196		2.641
19		2.682		1.619		2.603		2.398
20		1.726		1.538		1.725		7.280
21		1		1.098		1		4.178
22		1.029		2.288		1.032		1.758
23		1.122		1.038		1.116		1
24		1		1		1		1
25		1		1		1		1.892
26		1		1		1		2.507
27		1.039		4.368		1		4.983
28		1.008		1.002		1.009		1.017
29		1		1		1		1.020
30		1		1		1		1
31		1		1		1		1
32		1		1		1		1
33		1		1		1		1
34		1		1		1		1
35		1		1.412		1		1
I					0.1674	0.4332	0.2750	2.149
II					(all	0.2750	(all	1.268
III					flange	0.1001	flange	0.768
IV					variables	0.0262	variables	0.207
V					equal)	0.0214	equal)	0.008
VI						0.0285		0.256
$\bar{W}$	4.714	1.322	4.714	1.639	3.418	1.299	5.608	2.398

IV.2 Plate Containing Hole, under Uniform End Load

The design of a plate of length $3b$ and width $2b$ with a hole of diameter b at its center is now considered when a uniform normal in-plane loading is applied to the shorter edges, as illustrated in Fig. 4. The deformation of the plate is to be purely

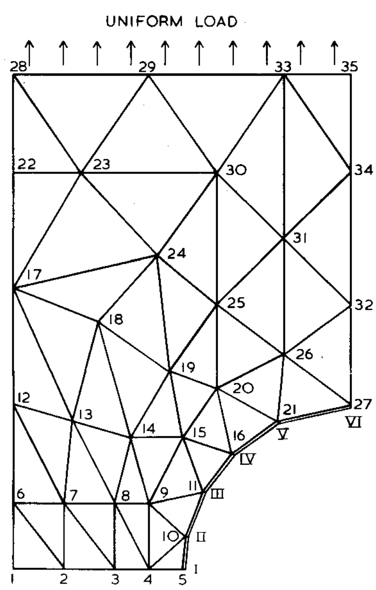

Fig. 5 Plate with hole—idealization with design variables numbered.

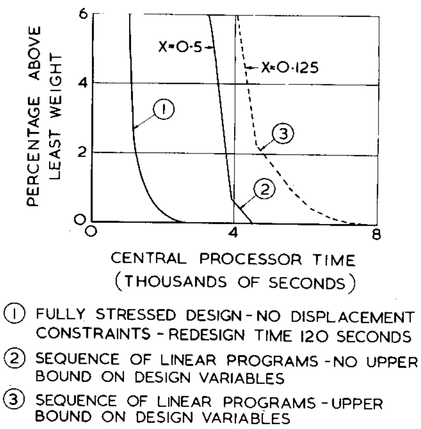

Fig. 6 Plate with unreinforced hole—computing times.

elastic and yielding is governed by the von Mises criterion. Four variants of the problem are studied:

Case 1. The optimum thickness distribution is sought when there is no upper limit on the local thickness, when there is no reinforcing member around the hole and when no displacement constraints are imposed.

Case 2. Same as Case 1 except that the local thickness is not permitted to exceed that of the thinnest plate of uniform thickness capable of carrying the loading without yielding.

Case 3. Same as Case 1 except that a reinforcing member of varying cross section is introduced around the hole.

Case 4. Same as Case 3 except that the relative displacements of opposite edges of the hole along the longitudinal and transverse centerlines of the plate must not exceed a specified absolute value. This value was so chosen that the corresponding "strain" would be equal in absolute value to 70% of the longitudinal strain in a least weight plate under the same loading in which there were no hole.

In all cases the minimum permissible thickness t_0 was taken to be that of the hypothetical plate without a hole referred to above under Case 4. The weights of the various designs are expressed in Table 2 in terms of a nondimensional parameter $\bar{W}$ which indicates the ratio of the weight to that of this hypothetical sheet.

Since both the plate and the loading are doubly-symmetrical, only one quadrant need be considered. The finite element idealization employed, which is shown in Fig. 5 involves 218

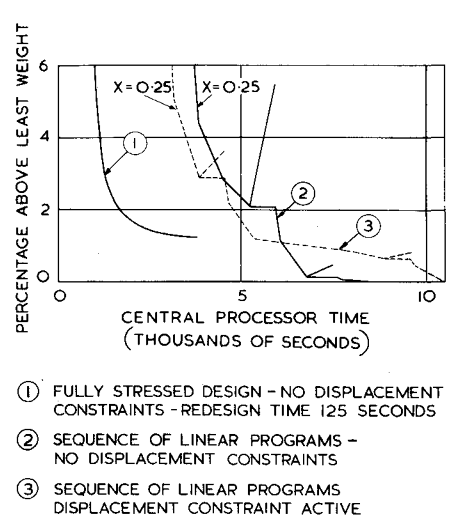

Fig. 7 Plate with reinforced hole—computing times.

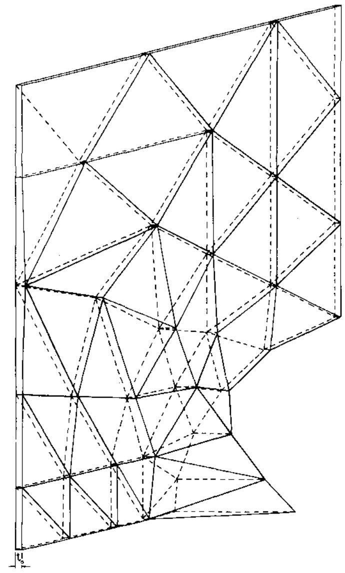

Fig. 8 Optimum plate with unreinforced hole—no upper bound on thickness.

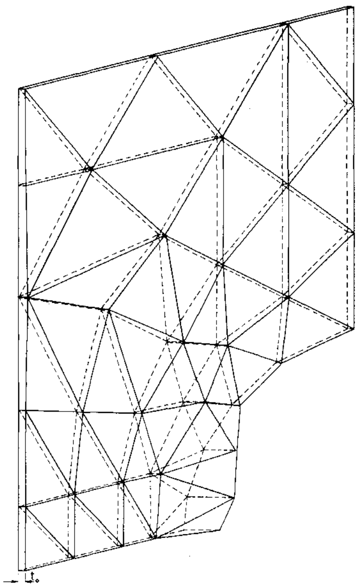

Fig. 10 Optimum plate with reinforced hole—no constraint imposed on displacements.

analysis variables, excluding those that are prescribed zero on account of symmetry. The thickness of the plane stress elements is specified by 35 design variables and the cross-sectional area of the reinforcing member, when present, by an additional 6 variables. Cases 1–4 involve 70, 70, 82, and 84 constraints, respectively; note that the upper bounds introduced on variables in Case 2 give rise to no additional constraints since, in effect, they replace move limits. The uniform loading is represented by point loads in the same manner as in the beam example.

The optimum designs computed are given in Table 2 in digital form and they are portrayed graphically in Figs. 8–12. Convergence properties are illustrated in Figs. 6 and 7. In cases 1 and 2 computations were initiated from the thinnest plate of uniform thickness which with the hole present, would carry the specified loading without yielding; the initial design in cases 3 and 4 incorporated a reinforcing member of arbitrary uniform cross-sectional area around the hole in a plate of uniform thickness.

In case 1 the designs obtained by the move limit method and by a fully-stressed design iteration were virtually identical. The significance of the solution is marred by a very large build-up in thickness adjacent to the hole which militates against the basic assumption of a plane distribution of stress; a similar result has been reported by Karnes and Tocher.[17] This build-up is not altogether surprising since, in a continuous analysis of the "least weight" thickness distribution around a circular hole in an infinite sheet under all-round tension,[18] the optimum solution requires a compact edge member adjacent to the hole. The unrealistic thickness distribution near the hole demonstrates how constraints which may not be anticipated at the problem formulation stage may need to be imposed before a meaningful solution is obtained.

Although case 2 involves no more variables or constraints than case 1 it is apparent from Fig. 6 that the former requires significantly more computing time on account of the active upper limits on six of the design variables (see Fig. 9). It should be noted that the finite element idealization is not strictly adequate to define the maximum thickness zone in the immediate vicinity of the hole. The thickening of the plate at the edge of the hole on the longitudinal centerline is presumably due to the influence of Poisson's ratio; a similar effect which has been observed in

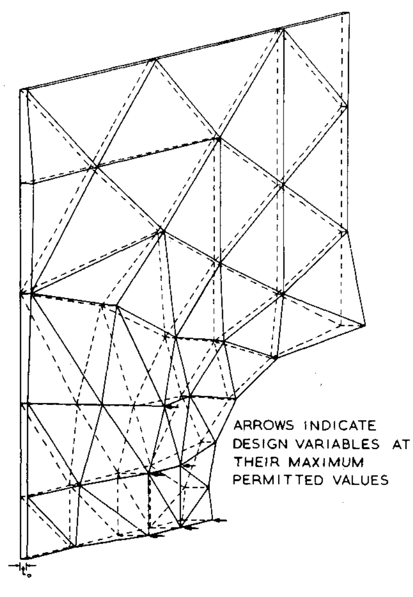

Fig. 9 Optimum plate with unreinforced hole—upper bound imposed on thickness.

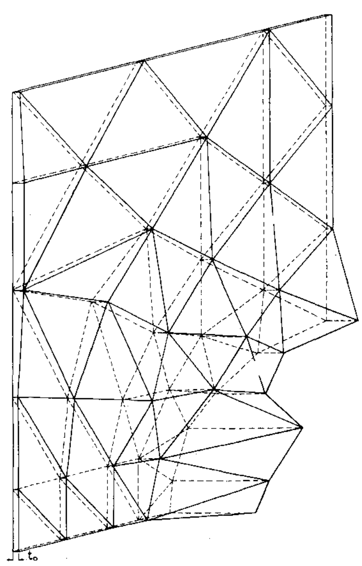

Fig. 11 Optimum plate with reinforced hole—constraint imposed on displacements.

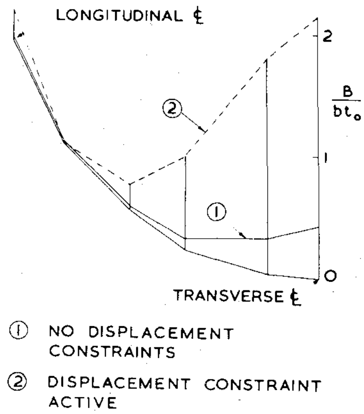

Fig. 12 Cross-sectional area of optimum flange around hole.

the variation in cross-sectional area of discrete reinforcement around neutral holes in uniform sheet[19] can be firmly attributed to this source.

The least weight design obtained in case 2 proved to be of the order of 1.2% lighter than that obtained by a fully-stressed design iteration. A difference between these designs would again be expected here for reasons similar to those discussed in Sec. IV.1. Note that in the presence of a reinforcing member around the hole, the very large local build-up in thickness observed in case 1 does not occur. The thickness in the vicinity of the hole and the cross-sectional area of the edge member are both somewhat excessive from the practical viewpoint in the optimum solution for case 4. In this design the constraint on the hole diameter parallel to the longer sides of the plate was active but that on the transverse diameter was slack (the optimum design for case 3 would violate both of these constraints).

V. Conclusions

It has been demonstrated that the move limit method is a viable method for use in the sizing of members in finite element idealizations of structures of fixed over-all geometry. This method is based directly on general nonlinear programing techniques and it remains applicable in problems which are beyond the scope of the rigorous application of optimality criterion methods. Least weight designs obtained using it in two examples proved to be a per cent or so lighter than corresponding designs obtained by a fully-stressed design iteration; larger percentage differences might well occur in situations where the load paths are necessarily complex and where several loadings are involved, e.g., around undercarriage bays. Convergence starting from an arbitrary initial design took about twice as much computing time in these examples as would be needed for a fully-stressed design iteration, the extra time being expended mainly in locating the vicinity of the optimum design. In some applications there would be merit in using the fully-stressed design approach to obtain a starting point reasonably near to the optimum design before embarking on a more rigorous search with the move limit method; this procedure should, however, be used with caution since the fully-stressed design may, on occasion, be close to a local rather than a global optimum. The convergence of the move limit method is not seriously retarded when active constraints are imposed on displacements as well as on stresses and it is in such applications that the method is likely to prove particularly useful.

References

[1] Pope, G. G. and Schmit, L. A., eds., "Structural Design Applications of Mathematical Programming Techniques," AGARDograph 149, Feb. 1971.

[2] Prager, W., "Conditions for Structural Optimality," *Computers and Structures*, Vol. 2, 1972, pp. 833–840.

[3] Taig, I. C. and Kerr, R. I., "Optimisation of Aircraft Structures with Multiple Stiffness Requirements," *Proceedings of the AGARD Second Symposium on Structural Optimisation*, Milan, Italy, April 1973, pp. 16–1 to 16–13, to be published.

[4] Venkayya, V. B., Knot, N. S., and Reddy, V. S., "Energy Distribution in an Optimum Structural Design," AFFDL-TR-68-156, March 1969, Air Force Flight Dynamics Lab., Wright-Patterson Air Force Base, Ohio.

[5] Gellatly, R. A. and Berke, L., "Optimal Structural Design," AFFDL-TR-70-165, Feb. 1971, Air Force Flight Dynamics Lab., Wright-Patterson Air Force Base, Ohio.

[6] Griffith, R. E. and Stewart, R. A., "A Non-Linear Programming Technique for the Optimization of Continuous Processing Systems," *Management Science*, Vol. 7, 1961, pp. 379–392.

[7] Reinschmidt, K. F., Cornell, C. A., and Brotchie, J. F., "Iterative Design and Structural Optimization," *Journal of the Structural Division, ASCE*, Vol. 92, No. ST6, 1966, pp. 281–318.

[8] Moses, F., "Some Notes and Ideas on Mathematical Programming Methods for Structural Optimization," Middelelse SKB 11/M8, Norges Tekniske Høgskole, Trondheim, Norway, Jan. 1967.

[9] Pope, G. G., "The Design of Optimum Structures of Specified Basic Configuration," *International Journal of Mechanical Sciences*, Vol. 10, 1968, pp. 251–263.

[10] Romstad, K. M. and Wang, C. K., "Optimum Design of Framed Structures," *Journal of the Structural Division, ASCE*, Vol. 94, No. ST12, 1968, pp. 2817–2845.

[11] Pope, G. G., "The Application of Linear Programming Techniques in the Design of Optimum Structures," *Proceedings of the AGARD Symposium on Structural Optimization*, Istanbul, Turkey, Oct. 1969, AGARD-CP-36-70.

[12] Moses, F., "Optimum Structural Design Using Linear Programming," *Journal of the Structural Division, ASCE*, Vol. 90, ST6, 1964, pp. 89–104.

[13] Hill, R., *The Mathematical Theory of Plasticity*, 1st Ed., Clarendon Press, Oxford, England, 1950.

[14] Argyris, J. H., "Triangular Elements with Linearly Varying Strain for the Matrix Displacement Method," *Journal of the Royal Aeronautical Society*, Vol. 69, 1965, pp. 711–713.

[15] Argyris, J. H., "Reinforced Fields of Triangular Elements with Linearly Varying Strain; Effect of Initial Strains," *Journal of the Royal Aeronautical Society*, Vol. 69, 1965, pp. 799–801.

[16] Argyris, J. H., "A Tapered TRIM 6 Element for the Matrix Displacement Method," *Journal of the Royal Aeronautical Society*, Vol. 70, 1966, pp. 1040–1043.

[17] Karnes, R. N. and Tocher, J. L., "Automatic Design of Optimum Hole Reinforcement," Boeing Rept. D6-23359, June 1968, The Boeing Co., Seattle, Wash.

[18] Mansfield, E. H., "Optimum Variable Thickness Reinforcement around a Circular Hole in a Flat Sheet under Radial Tension," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 24, 1971, pp. 499–507.

[19] Mansfield, E. H., "Neutral Holes in Plane Sheet—Reinforced Holes Which Are Elastically Equivalent to the Uncut Sheet," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 6, 1953, pp. 370–378.